Journal of Geoscience Education, v. 49, n. 5, p. 470-479, Nov. 2001 (edits, June 2005)

Computational Geology 18

Definition and the Concept of Set

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Topics this issue-
Mathematics: Predicate logic; predicate functions; relations.
Geology: Planets and moons; heliocentric orbits.

Prerequisite: CG-10, "The Algebra of Deduction", Mar 2000 (particularly: ~, ∧, →, and ↔).

Introduction

Geology involves measurement, calculation and modeling. Everyone would agree that these are mathematical activities.

Geology also involves classification. "In geology we tend to classify everything" (Thomas and Thomas, 2000, p. 571). This observation is certainly borne out by the GeoRef database. On 29 June, 2001, a search under the keyword "classification" brought up 60 papers with publication dates in 2001 and subjects ranging from stratigraphic units (Walsh, 2001) to pyroxenes (Yavuz, 2001).

Although we may not think of it as such, classification is a mathematical activity. The mathematics that is relevant to the kind of classification that underpins geologic terminology is included under set theory. The purpose of this and the next column is to illustrate that connection.

Aristotelian Definitions

The most basic classification is definition. In definition, a distinction is made between the "defined-in" class (the defined term), and the "defined-out" class (what the term intends to exclude).

Historically, definition as a formal activity marked the start of logic, which is associated with Aristotle (384-322 B.C.). Here is one account (Durant, 1961, p. 58-59):

> … Aristotle … almost without predecessors, almost entirely by his own hard thinking …created a new science – Logic….
>
> There was a hint of this new science in Socrates' maddening insistence on definitions, and in Plato's constant refining of every concept. Aristotle's little treatise on *Definitions* shows how his logic found nourishment at this source. 'If you wish to converse with me,' said Voltaire, 'define your terms.' How many a debate would have been deflated into a paragraph if the disputants had dared to define their terms! This is the alpha and omega of logic, the heart and soul of it, that every important term in serious discourse shall be subjected to strictest

> scrutiny and definition. It is difficult, and ruthlessly tests the mind; but once done it is half of any task.
>
> How shall we proceed to define an object or term? Aristotle answers that every good definition has two parts, stands on two feet: first, it assigns the object in question to a class or group whose general characteristics are also its own -- so man is, first of all, an animal; and secondly, it indicates wherein the object differs from all the other members in its class -- so man, in the Aristotelian system is a *rational* animal, his 'specific difference' is that unlike all other animals he is rational (here is the origin of a pretty legend). Aristotle drops an object into the ocean of its class, then takes it out all dripping with generic meaning, with the marks of its kind and group; while its individuality and difference shine out all the more clearly for this juxtaposition with other objects that resemble it so much and are so different.

As described by Durant, making an Aristotelian definition is a two-step process. In the first, the defined term is placed into a generic group. In the second, the defined term is distinguished specifically from other terms in that generic group. Thus the first step looks for similarities, and the second looks for differences. In the example cited by Durant, the generic group for the defined term "human" is "animal", and so the second step asks "How, specifically, do humans differ from all other animals?" Aristotle was not correct, of course, in his selection of a defining criterion. What is important is not *what* Aristotle thought, but rather *how* he thought.

Kirsten Peters (1996a) uses Aristotelian definitions as a way of combining short writing assignments with analytical thinking in the introductory geology classroom. Thus (Peters, 1996a, p. 65),

> On the first day of class, I explain that definitions are crucial to any technical field, that a well constructed definition reflects many different layers of understanding , and that to pass the final exam students will have to write several concise but complete definitions. The definitions must use a format we will practice many times during the course of the semester in weekly, short writing assignments.
>
> Following the spirit of Aristotle, I argue that understanding a term requires knowledge of other, broader ideas, to which the term is related. A definition, then, should begin with a clear first sentence which relates the term in question to a larger set of objects or ideas. Aristotle next recommends that we explain what the *distinguishing* features of the term are….

From my own experience using Aristotelian definitions for problem sets in geology courses, I agree with Peters' reference to "layers of understanding." Moreover, paraphrasing Durant, Aristotelian definitions are difficult, and they ruthlessly test the mind.

Define *Planet*

For a geologist, a reasonable place to start a discussion of definitions is, "Define *Earth*." As Peters (1996a) notes, the word "planet" immediately comes to mind for the generic group. In that case, a defining criterion for Earth can be "the third one" (counting outward from the Sun). Therefore, "The Earth is a planet that lies third from the Sun, between Venus and Mars" (Peters, 1996a, p 66) is sufficient to define Earth.

Defining Earth is not difficult. It does not test the mind. But what about the generic group that Peters uses for Earth. Does everyone agree on what a planet is?

According to the glossary in Skinner and Porter (1995, p. G11), *planet* means “a large celestial body that revolves around the Sun in an elliptical orbit.” We can impose an Aristotelian format on this definition and say that the generic group for planets is celestial bodies and the defining criteria are (1) large and (2) revolve around the Sun in elliptical orbits. What about these defining criteria? Do they allow in all the planets that we want to include and leave out all the non-planets that we want to exclude? We will take the two criteria one at a time.

Large. How large? Without quantification, this criterion is vague.

The largest asteroid, Ceres, has an equatorial radius (r_e) of 467 km (Beatty et al., 1999, appendix). Comets are smaller. We don't want asteroids or comets to be included among planets. So, at first glance, $r_e > 500$ km appears to be the place to start. That value will separate planets from asteroids and comets.

Unfortunately, $r_e > 500$ km will allow in 16 moons as well as the nine planets (Table 1). In fact, our Moon (1738 km) is larger than Pluto (1150 km). Ganymede (2634

Planet moon	Equatorial radius (km)	Mean distance from Sun (10^6km)	Mass (g)	Density (g/cm^3)	Mean distance from planet (10^6km)
Mercury	2,440	57.91	3.302×10^{26}	5.43	
Venus	6,052	108.21	4.865×10^{27}	5.20	
Earth	6,378	149.60	5.974×10^{27}	5.52	
Moon	1,738	149.60	7.349×10^{25}	3.34	0.384
Mars	3,396	227.94	6.419×10^{26}	3.91	
Jupiter	71,492	778.30	1.898×10^{30}	1.33	
Io	1,821	778.30	8.93×10^{25}	3.53	0.422
Europa	1,565	778.30	4.80×10^{25}	2.97	0.671
Ganymede	2,634	778.30	1.48×10^{26}	1.94	1.070
Callisto	2,403	778.30	1.08×10^{26}	1.85	1.883
Saturn	60,268	1,429.39	5.685×10^{29}	0.69	
Tethys	529	1,429.39	6.1×10^{23}	0.98	0.295
Dione	560	1,429.39	1.1×10^{24}	1.49	0.377
Rhea	764	1,429.39	2.3×10^{24}	1.24	0.527
Titan	2,575	1,429.39	1.34×10^{26}	1.88	1.222
Iapetus	720	1,429.39	1.6×10^{24}	1.0	3.561
Uranus	25,559	2,875.04	8.683×10^{28}	1.32	
Ariel	580	2,875.04	1.35×10^{24}	1.67	0.191
Umbriel	581×578	2,875.04	1.17×10^{24}	1.40	0.266
Titania	790	2,875.04	3.53×10^{24}	1.71	0.436
Oberon	760	2,875.04	3.01×10^{24}	1.63	0.583
Neptune	24,766	4,504.50	1.024×10^{29}	1.64	
Triton	1,353	4,504.50	2.15×10^{25}	2.05	0.355
Pluto	1,150	5,915.80	1.32×10^{25}	(2.0)	
Charon	625	5,915.80	1.6×10^{24}	(1.7)	0.0196

Table 1. Vital statistics of the 25 large (r_e>500 km) bodies orbiting the sun (Beatty et al., 1999, appendix).

E km) and Titan (2575 km) are larger than Mercury (2440 km). No number for "large" will, on its own, separate planets from moons of planets. That leaves "elliptical orbits about the Sun" to exclude large moons.

Elliptical orbits about the Sun. Galileo's *Dialogue Concerning the Two Chief World Systems* (1632) includes a drawing that exhibits the way we now think of planets and moons. The drawing (Fig. 1) shows the six planets known at the time orbiting the Sun; it also shows the Moon orbiting the Earth, and four moons (the ones he discovered) orbiting Jupiter. From this picture, the difference between planets and their moons seems perfectly clear: planets orbit the Sun, and moons orbit planets.

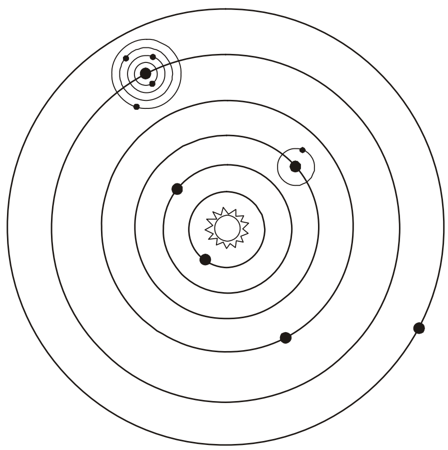

Figure 1. Concept of solar system as explained by Salviati to Simplicio and Sagredo on their third day of discussing Galileo's two world systems (drawing simplified from Galilei, 1632 [Drake, 1953], p. 323).

The problem with this picture is that it is drawn with two frames of reference: the planets are drawn from the point of view of the Sun, and the moons are drawn from the point of view of their respective planets. As a moon orbits a planet, and the planet orbits the Sun, the moon orbits the Sun. The planets' orbits are ellipses. The moons' orbits (around the Sun) average out to be the same ellipses. The heliocentric orbit of the Earth, for example, has an average radius of 149.6 million km, and the geocentric orbit of the Moon has an average radius of 384,000 km. The heliocentric orbit of the Moon also has an average radius of 149.6 million km (Table 1). "Elliptical orbits about the Sun" does not distinguish planets from moons.

If you don't believe it, think about drawing the Moon's heliocentric orbit to scale. Suppose you drew the Earth's orbit as a circle with a diameter of 15 cm to fit on this page. You would need to draw the path of the Moon within a band 0.4 mm wide straddling that circle. That is, you would need to lay out three circles: (1) a medial circle with radius 75.0 mm, (2) an inner circle with radius 74.8 mm, and (3) an outer circle with radius 75.2 mm. You would need to show the Moon's position fluctuating periodically from the outer circle to the inner circle as it moves along the band – passing in front of (i.e., on the Sun side of) the Earth 12.4 times as it completes the circle. You wouldn't see these differences between the Moon's and the Earth's heliocentric orbits, even if you could draw them! The departure of the Moon's path from that of the Earth's would be lost, practically, within the width of the line taken to draw the Earth's orbit.

The same point is illustrated in Figure 2 for circular orbits. The upper panel (A) shows 100 positions of a planet making a circular heliocentric orbit. The lower panel (B) shows 100 positions of a moon that (1) fluctuates about the planet within a band ± 2% of the heliocentric radius and (2) passes in front of the planet exactly five times (i.e., 5.0 synodic months in the planet's year). The question is, "Is the heliocentric orbit in the lower panel a circle?" I'm inclined to call it a slightly deformed (lopsided) circle. If that's the case, then, how deformed does a deformed circular (elliptical) orbit have to be before you cannot call it a circular (elliptical) orbit any more? The Moon's orbit, keep in mind, is much less deformed than that in Figure 2B. The Moon's band is ± 0.26% of the heliocentric radius, and there are 12.4 synodic months.

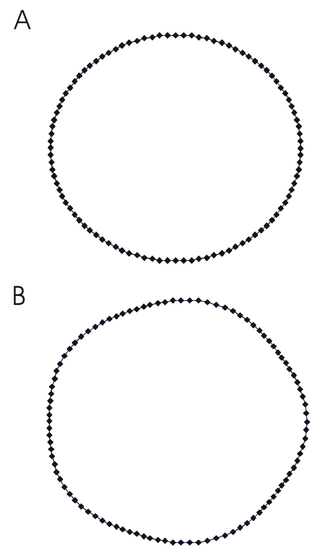

Figure 2. A: Circular heliocentric planetary orbit; *p* = 0 and *d* = 1 in Equation 1. B: Heliocentric orbit of a moon for which *p* = 6 and *d* = 50.

The Moon's orbit is like that of Figure 2B in that it is everywhere convex outward (Strahler, 1971, Fig. 8.3; Brannen, 2001). Io and Europa, on the other hand, make loops like the planet shown in Figure 3, and Ganymede and Titan make waves like the planet

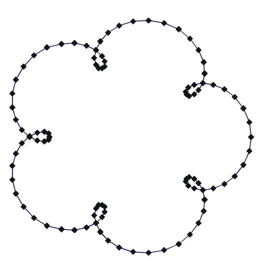

Figure 3. Heliocentric orbit of a moon with *p* = 6 and *d* = 4.

shown in Figure 4. The loops and waves of those actual moons are much smaller and much more numerous than those shown in Figures 3 and 4. In the case of Europa, for example, the width of the oribital band is ± 0.09% of the heliocentric radius (in contrast to 25% in Fig. 3), and there are more than 1000 synodic months per Jovian year (vs. five in Fig. 3). For Ganymede, the width of the orbital band is ± 0.14% of the heliocentric radius (in contrast to 6.7% in Fig. 4), and there are about 600 synodic months (vs. five in Fig. 4).

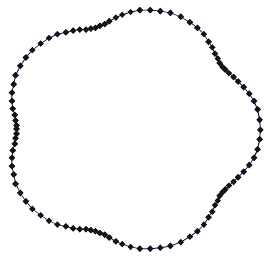

Figure 4. Heliocentric orbit of a moon with p = 6 and d = 15.

[The orbits of Figs. 2-4 were drawn using a spreadsheet based on an instructive paper published last summer in a journal of the Mathematics Association of America (Brannen, 2001). Parametric equations for the heliocentric orbit of a moon are

$$\begin{aligned} x(\theta) &= d\cos\theta + \cos p\theta \\ y(\theta) &= d\sin\theta + \sin p\theta\ , \end{aligned} \qquad (1)$$

where θ is the parameter, which ranges from $\theta = 0$ to $\theta = 2\pi$ radians for one complete orbit. The ratio of the Sun-planet distance to the planet-moon distance is d (the relative width of the orbital band used in the prior discussion is $1/d$ – 0.26% in the case of the Moon). The ratio of the planet's sidereal period to the moon's sidereal period is p (the number of synodic months is p–1). The figures were drawn by incrementing θ from 0 to 2π with steps of 0.02π, and then plotting y vs. x. Moons for which $d < p$ make loops. Moons with d between p and p^2 have wavy orbits. Moons with $d \geq p^2$ have orbits that are convex all around (Brannen, 2001).]

Distinction between Planets and their Moons

I raise the point about the Earth and Moon having nearly identical heliocentric orbits to illustrate how "layers of understanding" (Peters, 1996a) can come into play when one constructs Aristotelian definitions – not because I believe we have any trouble distinguishing between planets and moons. I use the words in Table 1, and I doubt that that causes any confusion. How, then, do we distinguish planets from moons if "large" and "orbit the Sun" don't work?

The discussion of heliocentric orbits of planets vs. moons is actually somewhat of a red herring. We all know from Physics 1 that it is the center of mass (barycenter) of a planet-moon pair that makes the elliptical orbit around the Sun; the planet and moon individually revolve around the center of mass. The important point is that, although the two form a pair by being gravitationally locked together as they both orbit the Sun, the planet is the planet, and the moon is the moon, because the former is larger than the latter.

A special case arises when the planet and moon are about the same size. For example, if the planet and moon were of equal size so that their center of mass was located midway between them, we would call the pair a binary planet. But what if one were twice the size of the other? Where do you draw the line?

One way of drawing the line is to require the center of mass of a planet-moon pair to lie outside of the planet (the larger of the two) for the pair to qualify as a binary planet. This criterion is easily calculated from the information in Table 1 and

$$s_b = \frac{\sum_i m_i s_i}{\sum_i s_i}, \qquad (2)$$

where s_b is the distance of the center of mass from the center of the planet, i refers to the members of the planetary group (i.e., the planet and its moons), and m_i and s_i refer to their masses and their distances from the center of the largest member, respectively. For the cases of Jupiter, Saturn, Uranus and Neptune, Equation 2 assumes that all the small bodies are lined up on one side of the largest body. Even so, the center of mass for each of these groups is only a very small percentage of the planet's radius away from the center (0.6%, 0.5%, 0.2% and 0.1%, respectively). For the Earth-Moon pair, however, the center of mass is 73% along the Earth's radius (about 1700 km below the Earth's surface, or within the lower mantle). For the Pluto-Charon pair, the center of mass is 185% of Pluto's radius from the center of Pluto (according to these data). Perhaps, then, we should think of Pluto and Charon as a binary planet (Stern and Mitton, 1999).

Another important feature of planets is that they are separated by vast amounts of space. This feature is captured in the following analogy (Stern and Mitton, 1999, p. 27):

> … The central, overarching aspect of the planetary system, which we all so casually overlook, is that it is almost completely empty. The planets themselves are but specks, spaced across millions and even billions of miles of nothingness. If the Earth were reduced in size to a simple, blue basketball, the sun (itself then about 100 feet in diameter) would lie about 12,000 feet away, with nothing but the little orbs Venus and Mercury occupying the space between the two. In this miniature, Jupiter, about 12 feet in diameter, would circle the Sun 10 miles away, followed by Saturn 20 miles out, Uranus some 40 miles out, and lonely Neptune, 60 miles distant from the tiny yellow hearth, glowing down below. Thus, within the [4000 square miles] corralled by Neptune's orbit in our Tinker-toy™ model, there is nothing but [a few] balloon-sized planets, their sun, and their tinier-still retinue of satellites, as well as a few-thousand scale-model asteroids (most no larger than sand grains), and perhaps a few-hundred comets, each, like the asteroids, barely a grit of sand against the yawning, empty vacuum. Empty upon Empty.

All these features – minimum size; distance of separation; size rank within a planetary cluster; location of the center of mass – can be expressed in words in the following statements:

Definition of planet: A planet is a celestial body that orbits the Sun, has an equatorial radius larger than 1000 km, and is larger than any other celestial body permanently within 30 million miles that also is larger than 1000 km and orbits the Sun.

Definition of a double planet: A double planet consists of two celestial bodies, one of which is larger than 1000 km, that orbit a common center of mass, which orbits the Sun and lies in the open space between the two celestial bodies.

Further information: Planets are separated by a wide expanse of space: tens of millions of kilometers for planets within 300 million kilometers of the Sun; hundreds of millions and even billions of kilometers for planets at greater distances.

Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto are all planets according to this definition. Asteroids, comets, moons and known members of the Kuiper belt are all excluded. Pluto-Charon is a binary planet.

The quantities used in the definition of planet are selected *ad hoc* from Table 1 to get the job done; they are not based on any understanding of fundamentals and, therefore, the definition, although it uses quantitative criteria, is hardly "scientific." The point of this column, however, is something different: that definition and the two other statements are tangles of words. All three statements can be said more clearly in the symbolic language of sets.

Sets and Membership in Sets

The symbolic expression, $x \in S$, says that x is an element of the set S. Similarly, $y \notin S$ says that y is not an element of S. Thus, if P is the set of planets, Earth $\in P$, and Io $\notin P$.

A set is most easily stipulated by listing all of its elements. For example, the easiest way of stipulating P as the set of planets is:

$$P = \{\text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Neptune, Pluto}\} \qquad (3a)$$

(note the commas and braces). The order of the elements in the set does not matter. Thus,

$$P = \{\text{Earth, Pluto, Jupiter, Mars, Mercury, Saturn, Venus, Neptune}\} \qquad (3b)$$

denotes the same set.

Another way of defining a set is to state *one or more properties* that characterize the elements of the set. Thus,

$$S = \{x \mid \Pi(x)\} \qquad (4)$$

says that S consists of all x possessing the property Π. The vertical bar is commonly read "such that."

The expression $\Pi(x)$ in Equation 4 includes the function, Π, and the variable, x. Π is a *predicate function* (as in "subject" and "predicate" of school grammar). As an example of a predicate function, let G be "was known to Galileo."

If G is a function, it must have a domain. A definition such as Equation 4, therefore, must make reference to a *domain of discourse*. For example, suppose we wish to define P_G as the set of planets that were known to Galileo. We can do this by stipulating the domain of discourse as planets (P) together with the function (G):

$$P_G = \{x \mid (x \in P) \wedge G(x)\}. \tag{5}$$

Equation 5, therefore, says that the set P_G consists of all x such that x is a planet and x was known to Galileo. Alternatively (and more compactly), we can write

$$P_G = \{x \in P \mid G(x)\}. \tag{6}$$

Without some such statement limiting the domain of discourse to planets, $\{x \mid G(x)\}$ stipulates a much larger set than we had in mind.

Thinking of $G(x)$ as a function means that we need to think about its range as well as its domain. Given that the domain is P, the range of the function $G(x)$ comprises the nine substitution instances: G(Mercury), G(Venus), G(Earth), G(Mars), G(Jupiter), G(Saturn), G(Uranus), G(Neptune), and G(Pluto). Each of these is a statement; G(Mercury), for example, is the statement "Mercury was known to Galileo." Each of these statements has a truth-value, T or F. For the first six (Mercury through Saturn), $G(x)$ is T. For the last three (Uranus through Pluto), $G(x)$ is F. Thus Equation 6 says that P_G consists of all x in P such that $G(x)$ is T.

Another way of looking at Equation 6 is that it defines P_G as the *truth-set* of P identified by $G(x)$. A truth-set consists of the members of the domain for which the predicate function produces a true statement.

As other examples, if $A(x)$ is "has an atmosphere", and $M(x)$ is "has one or more moons," then

$$P_a = \{x \in P \mid A(x)\} \tag{7a}$$

is the set of planets that have an atmosphere;

$$P_m = \{x \in P \mid M(x)\} \tag{7b}$$

is the set of planets that have one or more moons; and

$$P_{m,a} = \{x \in P \mid M(x) \wedge A(x)\} \tag{7c}$$

is the set of planets that have both one or more moons and an atmosphere.

Predicates can usefully refer to quantitative properties. For example, let $n_m(x)$ refer to the number of moons of x. Then,

$$P_m = \{x \in P \mid \ [n_m(x) \geq 1]\}, \tag{8a}$$

is an alternative expression for Equation 7b. As another example, let $\rho(x)$ be the density of x. Then,

$$P_t = \{x \in P \mid [\rho(x) > 3.0 \text{ g/cc}]\} \tag{8b}$$

specifies the planets that have a density larger than 3.0 g/cc ("terrestrial planets").

[I will use capital letters for complete predicate functions such as in Equations 7, and lower-case letters to indicate quantitative properties that are part of written out predicate functions enclosed in brackets as in Equations 8.]

Now, here's the connection: Equations 7 and 8 are Aristotelian definitions. When you construct an Aristotelian definition, you stipulate a set:

$$S = \{x \in D \mid (\Pi_1(x) \wedge \Pi_2(x)\} \tag{9}$$

The so-called "generic group" of the Aristotelian definition is the domain of discourse (D). The "defining criteria" are expressed by the predicate functions (Πs). Specifically, the defining criteria are that the substitution instances of the predicate functions are all true. An Aristotelian definition, therefore, defines a truth set in D.

Equality of Sets

Two sets are said to be equal if they contain exactly the same elements. Thus,

$$\{\text{Mercury, Venus, Earth, Mars}\} = \{x \in P \mid \rho(x) > 3.0 \text{ g/cc}\} \tag{10}$$

denotes the same set, namely P_t, the terrestrial planets.

Introductory logic books commonly make a distinction between *extensional* and *intensional* definitions. As stated by Salmon (1995, p. 52), "The extension of a term is the set of individuals, objects, or events to which the term can be correctly applied". In contrast (p. 53), "The intension of a term is the set of all and only those properties that a thing must possess for that term to apply to it." The left-hand side of Equation 10 is a complete extensional definition of terrestrial planets. The right-hand side is an intensional definition.

Complete extensional definitions are always unambiguous. In most cases, however, extensional definitions cannot be *complete* because the set is too large to enumerate. In such cases, partial – or representative – enumeration can be used, but this can introduce ambiguity.

The theme of this column is finding an intensional definition of planet equivalent to the extensional definition given in Equation 3. This requires a generalization from the kind of predicates considered in Equations 7 and 8.

Relations

The predicate functions of Equations 7 and 8 are functions of the single variable x. One can have multivariable predicate functions. Of importance to us is the predicate function of two variables, $R(x,y)$. The R stands for *Relation*, and the function states a relationship between the two variables, x and y.

The domain of $R(x,y)$ is the *product set* of the sets from which x and y are drawn. If x, for example, is drawn from set D_1 (i.e., $x \in D_1$), and y is drawn from set D_2 (i.e., $y \in D_2$), then the domain of $R(x,y)$ is the product set $D_1 \times D_2$. The product set consists of all the ordered pairs (x,y) made by pairing off all the x-values with all of the y-values in the order of, first, x, and second, y.

As an example, define

$$D_1 = \{\text{Moon, Io, Europa, Charon}\} \tag{11a}$$
$$D_2 = \{\text{Earth, Jupiter, Pluto}\} \tag{11b}$$

Then the product set consists of the twelve ordered pairs

$$D_1 \times D_2 = \{(\text{Moon, Earth}), (\text{Moon, Jupiter}), (\text{Moon, Pluto}), (\text{Io, Earth}), (\text{Io, Jupiter}), (\text{Io, Pluto}), (\text{Europa, Earth}), (\text{Europa, Jupiter}), (\text{Europa, Pluto}), (\text{Charon, Earth}), (\text{Charon, Jupiter}), (\text{Charon, Pluto})\}. \tag{12}$$

Imagine a spreadsheet table with the four elements of D_1 listed as labels in Column A, and the three elements of D_2 listed as labels across Row 1. Then, in the twelve cells composing the Block B2 to D5, write the ordered pair as (row label, column label). That is the product set in this case.

As an example rule for $R(x,y)$, consider the function $M(x,y)$ for "x is a moon of y". For the ordered pair (Moon, Earth), $M(x,y)$ is T. For the ordered pair (Io, Pluto), $M(x,y)$ is F. The truth-values for all members of $D_1 \times D_2$ are shown in the matrix of Figure 5. The truth-set of $M(x,y)$, then, consists of the four ordered pairs with the value T. Symbolically, we can write:

$$\{(\text{Moon, Earth}), (\text{Io, Jupiter}), (\text{Europa, Jupiter}), (\text{Charon, Pluto})\} = \{(x,y) \in D_1 \times D_2 \mid M(x,y)\}. \tag{13}$$

The left-hand side is an extensional definition of a set consisting of four moon-planet pairs. The right-hand side is an intensional definition of the same set.

	A	B	C	D
1		Earth	Jupiter	Pluto
2	Moon	T	F	F
3	Io	F	T	F
4	Europa	F	T	F
5	Charon	F	F	T

Figure 5. Matrix of truth-values for the function $M(x,y)$ and the domain $D_1 \times D_2$ defined by Equations 11. The truth set consists of the elements with T and is spelled out in Equation 13.

Whereas the example of Figure 5 relates the elements of one set to another, $R(x,y)$ can be used to compare elements within a single set. In such cases, x and y are both drawn from the same set, D. As an example, let the domain D for x and y be P_t, the four terrestrial planets. The domain for the relation $R(x,y)$ is the product set $D \times D$ (typically written D^2), consisting of 16 ordered pairs. Now, for $R(x,y)$, consider the function $L(x,y)$ for "x is larger than y" (meaning the equatorial radius of x is larger than the equatorial radius of y). The truthset consists of 6 ordered pairs within P_t^2 (Fig. 6). An intensional

	A	B	C	D	E
1		Mercury	Venus	Earth	Mars
2	Mercury	F	F	F	F
3	Venus	T	F	F	T
4	Earth	T	T	F	T
5	Mars	T	F	F	F

Figure 6. Matrix of truth-values for the function $r_e(x) > r_e(y)$ and the domain P_t^2, illustrating Equations 14 and 15.

definition of this set of ordered pairs can be written as

$$\{(x,y) \in P_t^{\,2} \mid L(x,y)\}. \tag{14}$$

Given the meaning of $L(x,y)$, it is clearer to say

$$\{(x,y) \in P_t^{\,2} \mid r_e(x) > r_e(y)\}. \tag{15}$$

Quantifiers

There are two quantifiers:

1. The universal quantifier, $\forall x$, which can be read "for all x", or "for any x".
2. The existential quantifier, $\exists x$, which can be read "there exists an x such that."

Quantifiers are combined with predicate functions to make propositions. For example, if the domain of x is the set P (planets), and the predicate function is $M(x)$ ("has one or more moons"), then $\forall x\ M(x)$ is the false proposition, "All planets have one or more moons," and $\exists x\ M(x)$ is the true proposition, "At least one planet has one or more moons." Other examples are the false proposition $\forall x \sim M(x)$ for "Every planet does not satisfy the criterion of having one or more moons", and the true proposition $\exists x \sim M(x)$ for "At least one planet does not satisfy the criterion of having one or more moons."

Quantifiers are commonly used to make compound propositions. For example, if the domain of x is P, and $A(x)$ is the proposition "has an atmosphere," then the statement

$$\forall x\ [M(x) \rightarrow A(x)] \tag{16}$$

is the true proposition that all planets that have one or more moons also have an atmosphere. As an example for the existential quantifier,

$$\exists x\ [\sim M(x) \wedge \sim A(x)] \tag{17}$$

is the true proposition that there is at least one planet that does not have a moon and does not have an atmosphere. (The planet is Mercury.)

A definition can be translated to a proposition by using the universal quantifier. For example, the definition of "Planets with moons" given in Equation 7b corresponds to the assertion

$$\forall x\ [(x \in P_m) \leftrightarrow [(x \in P) \wedge M(x)]], \qquad (18)$$

which says, for any x, if x is in P_m, it is a planet and has one or more moons (the left-to-right part), and it is in P_m if it is a planet and it has one or more moons (the right-to-left part). The right-to-left arrow says that all the correct things are included; the left-to-right arrow says that only correct things are included.

Multiple quantifiers are used with multivariable predicate functions (relations) to make *relational propositions*. For examples, suppose x is drawn from {Moon, Io, Europa, Charon} and y is drawn from {Earth, Jupiter, Pluto} (i.e., D_1 and D_2, respectively, of Equations 11). Further, let's use for $R(x,y)$, the relation $r_e(x) < r_e(y)$. Then,

$$\forall x\ \forall y\ [r_e(x) < r_e(y)] \qquad (19a)$$

is the false proposition that each of these four moons is smaller than each of these three planets;

$$\exists x\ \exists y\ [r_e(x) < r_e(y)] \qquad (19b)$$

is the true proposition that at least one of these moons is smaller than at least one of these planets);

$$\exists x\ \forall y\ [r_e(x) < r_e(y)] \qquad (19c)$$

is the true proposition that at least one of these moons is smaller than every one of these planets (i.e., Charon);

$$\forall y\ \exists x\ [r_e(x) < r_e(y)] \qquad (19d)$$

is the true proposition that for each of these planets, there is at least one of these moons that is smaller than the planet (e.g., the moon of each correct moon-planet pair is smaller than its planet);

$$\exists y\ \forall x\ [r_e(x) < r_e(y)] \qquad (19e)$$

is the true proposition that there exists one of these planets that is larger than each of these moons) (i.e., Jupiter);

$$\forall x\ \exists y\ [r_e(x) < r_e(y)] \qquad (19f)$$

is the true proposition that for each of these moons there is at least one larger planet (e.g., the planet of each correct moon-planet pair is larger than its moon). The statements

$$\forall y\ \forall x\ [r_e(x) < r_e(y)]$$

and

$$\exists y\ \exists x\ [r_e(x) < r_e(y)]$$

mean the same as 19a and 19b, respectively.

Multiple quantifiers and relations can be used to define sets. Following are four examples using D_1 and D_2 of Equations 11 again.

$$\{x \in D_1 \mid \exists y\ [(y \in D_2) \wedge [r_e(x) < r_e(y)]]\} \tag{20a}$$

selects from these four moons those that are smaller than at least one of these three planets. The set defined by 20a consists of all four moons in D_1.

$$\{x \in D_1 \mid \forall y\ [(y \in D_2) \rightarrow [r_e(x) < r_e(y)]]\} \tag{20b}$$

selects all the moons that are smaller than every one of the planets. This set consists of Charon only.

$$\{x \in D_1 \mid \exists y\ [(y \in D_2) \wedge [r_e(x) > r_e(y)]]\} \tag{20c}$$

selects the moons (Moon, Io, Europa) that are larger than at least one of the planets.

$$\{x \in D_1 \mid \sim\exists y\ [(y \in D_2) \wedge [r_e(x) < r_e(y)]]\} \tag{20d}$$

selects the moons for which there is not a larger planet. This set is denoted by $\varnothing$, the null (empty) set. ($\varnothing$ is a Scandinavian letter, not the Greek letter, phi.)

We can now return to the problem of defining "planet."

About Planets

Let D be the set of celestial bodies orbiting the Sun. Let $s(x,y)$ be the distance between x and y. Let $r_e(x)$ be the equatorial radius of (x), as before. Then, the set of planets in our solar system can be defined as:

$$\begin{aligned} P = \{x \in D) \mid [r_e(x) > 10^3 \text{ km}] \wedge \\ \forall y\ [[(y \in D) \wedge (y \neq x) \wedge \\ [s(x,y) < 10^7 \text{ km}]] \rightarrow [r_e(x) > r_e(y)]]\}. \end{aligned}$$

According to this definition, planets are, first of all, celestial bodies that orbit the Sun. Each planet has an equatorial radius larger than 1000 km. Finally, a planet has a larger equatorial radius than any other celestial body (y) within 10 million kilometers that is also orbiting the Sun. (The condition $y \neq x$ represents the word "other.")

To define the set of binary planets in our solar system (P_B), let $B(x,y)$ be the property that x and y orbit a common barycenter that orbits the Sun ("B" for barycenter), and let $s_b(x,y)$ be the distance of the barycenter of x and y from the center of x. Then

$$P_B = \{(x,y) \in P \times D \mid B(x,y) \wedge [s_b(x,y) > r_e(x)]\}.$$

According to this definition, a binary planet is an ordered pair consisting of a planet and a celestial body that orbits the Sun. The two bodies must orbit a common center of mass that orbits the Sun, and that center of mass must lie outside the planet. (The $y \neq x$ condition is not needed, because the center of mass of a planet with itself -- if one wants to consider such a thing -- would not lie outside of itself.) (Both this definition and the definition of planets break down for the case of two nearby planet-size bodies of precisely equal size -- but this case doesn't exist in our solar system, and so I won't complicate these statements by trying to allow for it.)

Finally, for the "empty-upon-empty" feature of planets in space, we can start with

$$\forall x \forall y\ [[((x,y) \in P^2\ ) \wedge (y \neq x)] \rightarrow [s(x,y) > 10^7 \text{ km}]].$$

This says that each planet is more than 10 million kilometers away from every other planet. To give more information, let $s(x)$ be distance from the Sun. Then

$$\forall x \forall y\ [[((x,y) \in P^2\ ) \wedge (y \neq x)] \rightarrow$$

$$[[[s(x) \leq 10^{8.5} \text{ km}] \rightarrow [s(x,y) > 10^7 \text{ km}]] \wedge$$

$$[[s(x) > 10^{8.5} \text{ km}] \rightarrow [s(x,y) > 10^8 \text{ km}]]]].$$

The first part says you are comparing each planet to every other planet. The second part says that if the first planet is within 300 million km of the Sun, the other planet is more than 10 million kilometers away. The third part says that if the first planet is more than 300 million kilometers away from the Sun, the other planet is more than 100 million kilometers away. The most difficult part of the exercise is keeping track of the brackets. This can be done by expanding them into nested rectangles to box appropriate segments.

Final Remark

Charles Sanders Peirce (1839-1914) was a geophysicist who worked for the U.S. Coast and Geodetic Survey for some 30 years. A member of the National Academy of Sciences, Peirce was internationally recognized in the 1870s and 1880s for his pioneer work in pendulum gravimetry. He is better known now as one of the foremost American philosophers of all time. According to Peirce, "… familiarity with a notion [is] the first step toward clearness of apprehension, and the defining of it the second."

Copi and Cohen (1998) use that quotation at the beginning of their chapter "Definition" in their *Introduction to Logic*, one of the standard textbooks on the subject. Within the chapter, they use *denotative* and *connotative* instead of "extensional" and "intensional," respectively, and *definition by genus and difference* instead of "Aristotelian definition." And, near the end, they say this (Copi and Cohen,1998, p. 152): “For most purposes, connotative definitions are greatly superior to denotative definitions; and of all connotative definitions, those constructed by genus and difference are usually most effective and most helpful when one is reasoning or engaging in other informative uses of language."

Putting these quotations together, we have it that defining terms promotes understanding, as well as communication, and that the best way of going about it is to use Aristotelian definitions. The trouble with this is that one can easily get tied up in the words. If you find that happening, remember that you are simply defining a set. Use symbols -- first, to identify the domain of discourse and, then, to represent the functions that express the defining criteria. That way you see through the tangle of words to the logic of your definition.

Suggested Reading

The article by Peters (1996b) reprises an appendix of her terrific little book, *No Stone Unturned* (Peters, 1996b). The book aims to introduce *geologic reasoning* to introductory geology students.

Another great read is *Pluto and Charon* (Stern and Mitton, 1999), a lively account of the anticipation, discovery and interpretation of Pluto (1930), Charon (1978) and the Kuiper Belt (1990s). The writing is wonderful: "Empty upon Empty… That's why they call it space" (p. 27); "Charon's Harvest" (p. 54); "Where Have All the Plutos Gone?" (p. 155).

The logic discussed in this column is *predicate logic* (or *predicate calculus*) and is included in most standard textbooks used in the introductory logic course taught in philosophy departments (e.g., Salmon, 1995; Copi and Cohen, 1998). The book I find most helpful is the brief, direct and clear introduction to mathematical logic by Stolyar (1970 [1983]). The subject is now included in a course relatively new in the mathematics curriculum: discrete mathematics. Johnsonbaugh (2001) is a standard textbook

Acknowledgments

I thank Steve Maurer at Swarthmore for instructive comments about the mathematics in an early draft, and Jeff Ryan at USF for introducing me to planets and moons.

References Cited

Beatty, J.K., Peterson, C.C., and Chaikin, A., 1999, The new solar system, 4th edition: Cambridge Univ. Press, 421 p.

Brannen, N.S., 2001, The Sun, the Moon, and convexity: The College Mathematics Journal, v. 32, p. 268-272.

Copi, I.M. and Cohen, C., 1998, Introduction to logic, 10th edition: Upper Saddle River, Prentice Hall, 714 p.

Durant, W., 1961, The story of philosophy: New York, Simon and Schuster, 543 p.

Galilei, Galileo, 1632, Dialogue concerning the two chief world systems -- Ptolemaic & Copernican, translated by Stillman Drake, foreword by Albert Einstein: Berkeley, Univ. California Press, 1953, 496 p.

Johnsonbaugh, R., 2001, Discrete mathematics, 5th edition: Upper Saddle River, NJ, Prentice Hall, 621 p.

Peters, E.K., 1996a, Writing across the curriculum meets introductory geology: Journal Geoscience Education, v. 44, p. 65-67.

Peters, E.K., 1996b, No stone unturned, Reasoning about rocks and fossils: New York, Freeman, 237 p.

Salmon, M.H., 1995, Introduction to logic and critical thinking, 3rd edition: Fort Worth, Harcourt Brace College Publishers, 471 p.

Skinner, B.J. and Porter, S.C., 1995, The dynamic Earth: An introduction to physical geology: New York, Wiley, 567+36 p.

Stern, A. and Mitton, J., 1999, Pluto and Charon, Ice words on the ragged edge of the solar system: New York, Wiley, 216 p.

Stolyar, A.A., 1970, Introduction to elementary mathematical logic: Cambridge, MIT Press (New York, Dover, 1983), 209 p.

Strahler, A.N., 1971, The Earth sciences, 2nd edition: New York, Harper & Row, 824 p.

Thomas, J.J. and Thomas, B.R., 2000, Classification: Journal of Geoscience Education, v. 48, p. 571.

Walsh, S.L., 2001, Notes on geochronologic and chronostratigraphic units: Geological Society of America Bulletin, v. 113, p. 704-713.

Yavuz, F., 2001, PYROX; a computer program for the IMA pyroxene classification and calculation scheme: Computers & Geosciences, v. 27, p. 97-107.